Journal of Sound and Vibration (2002) **254**(4), 763–775
doi:10.1006/jsvi.2001.4121, available online at http://www.idealibrary.com on IDEAL®

TIME INTEGRATION OF NON-LINEAR DYNAMIC EQUATIONS BY MEANS OF A DIRECT VARIATIONAL METHOD

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(*Received* 20 *April* 2000, *and in final form* 1 *November* 2001)

Non-linear dynamic problems governed by ordinary (ODE) or partial differential equations (PDE) are herein approached by means of an alternative methodology. A generalized solution named WEM by the authors and previously developed for boundary value problems, is applied to linear and non-linear equations. A simple transformation after selecting an arbitrary interval of interest T allows using WEM in initial conditions problems and others with both initial and boundary conditions. When dealing with the time variable, the methodology may be seen as a time integration scheme. The application of WEM leads to arbitrary precision results. It is shown that it lacks neither numerical damping nor chaos which were found to be present with the application of some of the time integration schemes most commonly used in standard finite element codes (e.g., methods of central difference, Newmark, Wilson-θ, and so on). Illustrations include the solution of two non-linear ODEs which govern the dynamics of a single-degree-of-freedom (s.d.o.f.) system of a mass and a spring with two different non-linear laws (cubic and hyperbolic tangent respectively). The third example is the application of WEM to the dynamic problem of a beam with non-linear springs at its ends and subjected to a dynamic load varying both in space and time, even with discontinuities, governed by a PDE. To handle systems of non-linear equations iterative algorithms are employed. The convergence of the iteration is achieved by taking n partitions of T. However, the values of T/n are, in general, several times larger than the usual Δt in other time integration techniques. The maximum error (measured as a percentage of the energy) is calculated for the first example and it is shown that WEM yields an acceptable level of errors even when rather large time steps are used.

1. INTRODUCTION

A generalized solution named whole element method (WEM) which has been previously developed and applied to boundary value (BV) problems [1–4] is used here to solve initial conditions (IC) and/or boundary conditions (BC) problems, linear or not, governed by ordinary or partial differential equations. Traditionally, the variational methods such as Ritz have been applied to the spatial domain in BV problems [5, 6]. When dealing with the

0022-460X/02/$35.00

time variable, the methodology may be seen as a time integration scheme. Its performance in solving a pair of non-linear ordinary differential equations is evaluated in comparison with other commonly used time integration schemes, which were previously assessed by Xie [7]. A previous work by Low [8] also reported a similar study. Also, another application is the solution of a non-linear initial conditions–boundary value (IC–BV) problem using the variational method in both the spatial and the temporal variables. In both cases, WEM behavior is excellent and it lacks neither numerical damping nor chaos, which may be present when some equations under certain conditions are solved, as will be shown. This is particularly true with some of the time integration schemes most commonly used in the standard finite element codes (extensive references on the subject may be seen for instance in reference [9]).

In this work, an alternative technique which arises from a direct variational method first developed by the authors for BV problems [1–4] is employed. Such a method may be extended to the solution of initial conditions problems governed by an ordinary differential equation (ODE) as reported in reference [10] and of IC–BV problems governed by partial differential equations (PDE) [11, 12]. The usage of this formulation as a time integration scheme is not traditional. For instance, in the relevant book by Reddy [13], the IC–BV problems are approached with the variational formulations in the space variable and the usual time integration techniques (central differences, Newmark, and so on) when dealing with the temporal variable. WEM, in this case, does not require very small steps to yield accurate results. In particular, when non-linear (though simple) differential equations are addressed, as shown in reference [7], some difficulties may arise with various time integration schemes. Instead, WEM yields convergent results for various IC even using rather large steps. The other schemes may lead to non-convergent solutions or give meaningless results with such steps. It should be noted that WEM represents not an approximation but a theoretic exact solution (arbitrary precision results) in each time step.

As mentioned before an application of WEM to an IC–BV problem is also shown: the forced vibration of beams with a dynamic distributed load discontinuous both in space and time and supported at its ends by non-linear springs. In this problem it was also possible to obtain convergent solutions.

2. TRANSFORMATION OF THE IC–BV PROBLEM IN A BV PROBLEM

Since the WEM method has been developed for BV [1–4] problems, a previous transformation [10–12] will be performed as follows. WEM makes uses of extremizing sequences, which linearly combine functions belonging to a complete set in the domain $D[0, 1]$ in n-dimensions. Then in problems with semi-infinite domain ($t \geqslant 0$), an arbitrary interval T ($T \geqslant t \geqslant 0$) of interest (in which the response is of interest) is introduced. A change of variable is now obtained as

$$\tau = t/T, \tag{1}$$

where $0 \leqslant \tau \leqslant 1$ is the non-dimensionalized time. Let us deal with an IC–BV problem governed by the PDE of solution $\hat{u} = \hat{u}(x, t)$:

$$A^* \hat{u} = f^*(x, t) \quad \text{in } \mathbf{D} \tag{2}$$

being $\{\mathbf{D}: 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$ (in one, two or three spatial dimensions), and the following conditions apply:

$$IC = \begin{cases} \hat{u}(x,0) = U_0(x), \\ \partial\hat{u}/\partial t(x,0) = V_0(x), \end{cases} \quad B.C. \quad \text{in } x = 0,\ 1. \tag{3, 4}$$

A^* is the differential operator, f^* is, at least, a square integrable function, x is the spatial non-dimensional variable, and t is time (in general). Now problem (2) with conditions (3) and (4) is transformed into

$$Au = f(x, \tau), \tag{5}$$

where $u = \hat{u}(x, t) = \hat{u}(x, T\tau) = u(x, \tau)$. In what follows, and without loss of generality, we will accept homogeneous BC. A is changed so that it takes into account that

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial u}{\partial \tau}\frac{1}{T}. \tag{6}$$

The IC are written as

$$IC = \begin{cases} u(x,0) = U_0(x), \\ \partial u/\partial\tau(x,0) = TV_0(x). \end{cases} \tag{7}$$

The problem then will be handled as a boundary value one, both in the spatial (x) and temporal (originally t, transformed in τ) variables so that WEM may be applied. Let us now introduce a function

$$\psi = \psi(x, \tau) = [U_1(x) - U_0(x)]\tau + U_0(x), \tag{8}$$

which is the most simple function that satisfied $\psi(x, 0) = U_0(x)$ and $\psi(x, 1) = U_1(x)$ and where $U_1 \equiv u(x, 1)$. Obviously, $U_0(x)$ is known with the problem statement but this is not the case with $U_1(x)$. With function (8) we define

$$w = w(x, \tau) = u(x, \tau) - \psi(x, \tau), \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant \tau \leqslant 1 \tag{9}$$

which verifies $w(x, 0) = w(x, 1) = 0$. In conclusion, we have transformed an IC–BV problem into a BV one with homogeneous BC. At the same time the differential equation is

$$A(w + \psi) = f(x, \tau). \tag{10}$$

If the problem has only IC (i.e., governed by an ODE), only variable t (τ after being transformed) will be involved and the problem is reduced to

$$\begin{aligned} &A(w + \psi) = f(\tau), \qquad w = w(\tau) = u(\tau) - \psi(\tau), \\ &\psi(\tau) = (U_1 - U_0)\tau + U_0, \qquad w(0) = w(1) = 0. \end{aligned} \tag{11}$$

The IC related to the first derivative is $\partial u/\partial\tau(0) = TV_0 = \partial w(0)/\partial\tau - (U_1 - U_0)$ from which the unknown U_1 is eliminated as $U_1 = U_0 - TV_0 + \partial w/\partial\tau(0)$.

3. WHOLE ELEMENT METHOD (WEM) DESCRIPTION

Since WEM is a direct variational method, the generalized solution of a boundary value problem arises from the extremization of an *ad hoc* functional. Among the features of this method it is worth mentioning the systematic statement of extremizing sequences for diverse problems and domains and the theoretical foundation of the methodology. Rosales has demonstrated [4] that the usual procedure of extremizing a functional is equivalent to finding the solution of equations of the type

$$[A(w_{MP} + \psi) - f, \delta w_{MP}] = 0, \tag{12}$$

using integration by parts whenever possible and where $(f, g) \equiv \int_0^1 f(\xi)\, g(\xi)\, \mathrm{d}\xi$ denotes an internal product in the space dimension of the problem. This equation is a *pseudo* virtual work statement using w_{MP} (the extremizing sequences used in WEM). Having the differential equation there is no need to state a functional. δ_{MP} denotes the first variation of w_{MP} w.r.t. the unknowns. After integration by parts one finds a statement that gives the tool for the practical application of WEM.

The extremizing sequences to be used in WEM are systematically stated in any dimension. It has been shown that only the essential conditions may be satisfied by the sequence (not in general by each co-ordinate function). The systematic generation procedure of these trigonometric extended series may be read in references [2, 4]. Let us just show an example of one of the infinite combinations that give rise to an extremizing sequence that verifies *uniform convergence* (UC) towards a continuous function $\phi(x, y)$ in the domain R^2 $\{D{:}0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$:

$$\phi_{MP}(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{P} A_{ij} s_i s_j + \underline{x\left(a_0 + \sum_{j=1}^{P} A_{0j} s_j\right) + y\left(b_0 + \sum_{i=1}^{M} A_{i0} s_i\right)}$$

$$\underline{+ A_{00}xy + \sum_{j=1}^{P} b_j s_j + \sum_{i=1}^{M} a_i s_i + k_0}, \tag{13}$$

where $s_i \equiv \sin \alpha_i x$, $s_j \equiv \sin \alpha_j y$, $\alpha_n = n\pi$ and with which

$$|\phi_{MP} - \phi| \to 0 \quad \text{as } M, \quad P \to \infty, \ \forall x, y \in D. \tag{14}$$

As may be observed the statement is not immediate. The underlined part may be interpreted as the (not apparent) enlargement of an elementary Fourier series with convergence in L_2. In an analogous way series for any dimensional domain may be generated.

Finally, the authors have demonstrated that using these series, along with eventual *Lagrangian* multipliers, the application of WEM yields uniform convergence solutions of the essential functions and exact eigenvalue. The theoretically exact results are found as arbitrary precision numbers in the numerical algorithm. Here we name essential functions as those involving functions of order $\leqslant k$, with $4k$ being the largest derivative in the differential equation (the authors have extended the methodology also for odd equations). For instance, in the problem of a vibrating beam, the displacement and the slope are essential functions.

4. APPLICATION OF WEM TO TWO NON-LINEAR IC PROBLEMS GOVERNED BY ODEs

Two illustrations of WEM application to non-linear ODEs will be shown in this section. Both may be thought of as modelling the dynamics of an s.d.o.f. system of a mass and spring with different non-linearity laws in each case. The equations are

$$\ddot{v} + \bar{k} \tanh v = 0, \qquad \ddot{v} + \bar{k}_1 v + \bar{k}_2 v^3 = 0 \tag{15, 16}$$

with $v = v(t)$, $0 \leqslant t < \infty$ and $\bar{k}$, $\bar{k}_1$, $\bar{k}_2$ are proportional to the spring elastic properties. Dots denote derivative with respect to the time variable. In each the IC, $v_0 \equiv v(0)$ and $\dot{v}_0 \equiv \dot{v}(0)$, are also given. Equation (16) is the well-known Duffing model without damping.

4.1. WEM SOLUTION FOR EQUATION (15)

The change of variable (1) and the introduction of a function ψ as suggested in equation (8) and in particular in equation (11) lead to the transformed equation and its BC:

$$u'' + k \tanh(u + \psi) = 0, \quad u(0) = u(1) = 0 \tag{17}$$

with $u = u(\tau)$ and $k \equiv \bar{k}T^2$. The prime denotes the derivative with respect to τ. Let us introduce the WEM sequence as sequences with uniform convergence

$$u'_M(\tau) = \sum_{i=1}^{M} A_i c_i + A_0, \qquad u_M(\tau) = \sum_{i=1}^{M} \frac{A_i s_i}{\beta_i} + A_0 x + B_0, \tag{18, 19}$$

where A_i are unknowns, $s_i \equiv \sin(\alpha_i \tau)$, $c_i = \cos(\alpha_i \tau)$ and $\alpha_i = i\pi$. The fulfillment of BC (17) yields A_0 and B_0 null. The application of statement (12) to equation (17) gives place to

$$-(u'_M, \delta u'_M) + k \tanh(u_M + \psi), \delta u_M = 0 \tag{20}$$

from which

$$A_i = 2kH_i, \tag{21}$$

where

$$H_i = [\tanh(u_M + \psi), s_i] \tag{22}$$

and $U_1 - U_0 = TV_0 - S_A$, $S_A = \sum_i A_i$. The value of T, the interval of interest, is arbitrary though fixed for each numerical experiment. In general, it is found convenient to solve equation (21) with an iteration algorithm thus avoiding the manipulation of rather large systems of non-linear equations. On the other hand, a limitation to the interval of interest T is introduced. In effect, such a limit exists above which T conduces to a divergent algorithm. So as to overcome this behavior, n partitions of T are taken. The values of T/n, in all the analyzed cases, are larger than the usual Δt of the most popular numerical integration schemes (e.g., method of central difference, Newmark method, Wilson-θ method, etc.). Let us show a numerical example by setting $\bar{k} = 100$ (softening spring) in equation (15) and the initial conditions $v_0 = 4$ and $\dot{v}_0 = 0$. This oscillator is known to have a period of

$$T_p = 4 \int_0^4 \frac{\mathrm{d}u}{\sqrt{661{\cdot}43 - 200 \ln[\cosh(u)]}}, \tag{23}$$

which yields $T_p = 1{\cdot}14$ s. The exact solution of u may be found using analytical methods as described in reference [14]. The authors have found it by using algebraic series. Xie [7] reported results found with various numerical schemes and most of them added strong

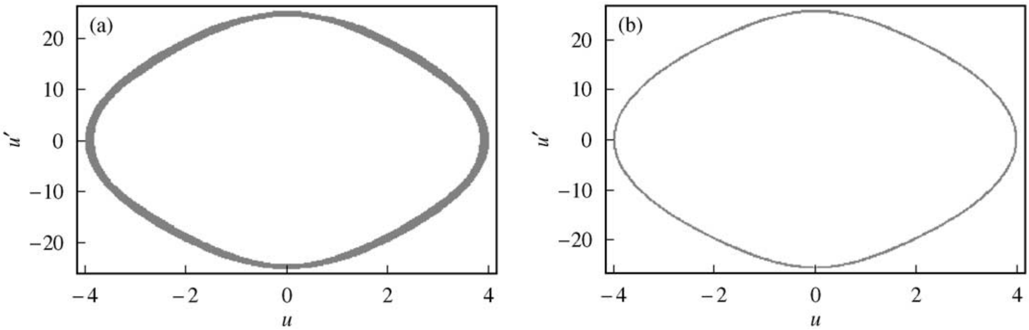

Figure 1. Phase plane of the equation $\ddot{v} + 100 \tanh v = 0$, $v_0 = 4$ and $\dot{v}_0 = 0$. Time duration of the experiment $T = 114$ s. (a) $M = 10$, $N = 10$, $T/n = 0{\cdot}25$ s, (b) $M = 20$, $N = 20$, $T/n = 0{\cdot}125$ s.

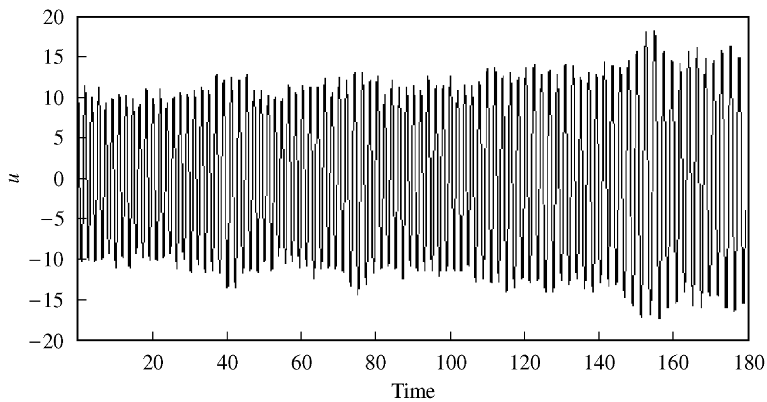

Figure 2. Time–displacement of equation $\ddot{v} + 100 \tanh v = 0$, $v_0 = 10$ and $\dot{v}_0 = 0$. $M = 10$, $N = 10$, $T/n = 0{\cdot}75$ s.

numerical damping when a step $\Delta t = T_p/20 = 0{\cdot}057$ s was used. Figure 1 shows the phase diagram found using WEM and a fairly large step $T/n = 0{\cdot}25$ s. In Figure 1(a), $M = 10$ (number of terms in the WEM series) and $N = 10$ (numerical integration terms for H_i). The result is further improved by taking $M = 20$. A similar result may also be achieved by decreasing the step to $0{\cdot}125$ s. The resulting phase plot is shown in Figure 1(b), in which $M = 20$, $N = 10$ and $T/n = 0{\cdot}125$ s.

It is known that some time integration schemes are unconditionally stable when dealing with linear problems. This is not the case when they are applied to non-linear equations. Xie [7] reported another example with $\bar{k} = 100$, $v_0 = 10$, $\dot{v}_0 = 0$, and a step $\Delta t = 0{\cdot}225$ s. Its period is $T_p = 1{\cdot}8$ s. Even the α-method, which proved to yield moderate numerical damping, renders an unstable solution (the amplitude raised to a value of 400 in 180 s). When WEM is applied to this case with $M = 10$, $N = 10$ it was necessary to increase the step to $T/n = 0{\cdot}75$ s. (40% of the period) in order to obtain a clue regarding non-stable behavior as shown in Figure 2 (the amplitude is less than 20 in 180 s).

4.2. WEM SOLUTION FOR EQUATION (16)

The application of WEM to equation (16) is totally analogous to the procedure described in section 4.1 above. Thus the time variable transformation and introduction of the function

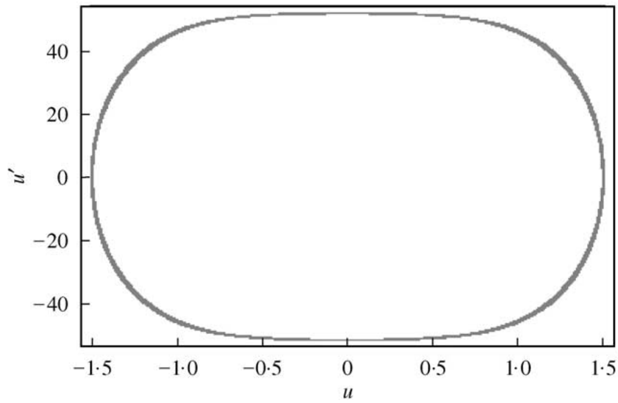

Figure 3. WEM solution of the phase plane of equation $\ddot{v} + 100v(1 + 10v^2) = 0$, $v_0 = 1{\cdot}5$ and $\dot{v}_0 = 0$. Time duration of the experiment $T = 15$ s. $M = 10$, $N = 10$, $T/n = 0{\cdot}02$ s.

ψ give rise to

$$u'' + k_1(u + \psi) + k_2(u + \psi)^3 = 0, \qquad u(0) = u(1) = 0, \tag{24}$$

with $u = u(\tau)$, $k_1 = \bar{k}_1 T^2$ and $k_2 = \bar{k}_2 T^2$. Let us introduce the WEM sequences (18) and (19). The application of statement (12) to equation (24) yields

$$-(u'_M, \delta u'_M) + [k_1(u_M + \psi) + k_2(u_M + \psi)^3], \quad \delta u_M = 0, \tag{25}$$

$$A_i = 2[(S_A - T\dot{v}_0)k_1 L_{i0} - k_1 v_0 L_{i1} - k_2 H_i], \tag{26}$$

where

$$S_A \equiv \sum_{i=1}^{M} A_i, \qquad L_{i0} = \frac{1 - (-1)^i}{\alpha_i^2}, \qquad L_{i1} = \frac{(-1)^{i+1}}{\alpha_i^2}, \qquad H_i \equiv \frac{[(u + \psi)^3, s_i]}{\alpha_i^2}.$$

Let us analyze equation (16) choosing $\bar{k}_1 = 100$, $\bar{k}_2 = 1000$ with $v_0 = 1{\cdot}5$ and $\dot{v}_0 = 0$. WEM was applied using $M = 10$, $N = 10$ and $T/n = 0{\cdot}02$ s. The period T_p may be found by solving

$$T_p = 4\int_0^{1\cdot5} \frac{\mathrm{d}u}{\sqrt{2756{\cdot}25 - 100u^2 - 500u^4}} = 0{\cdot}151 \text{ s}. \tag{27}$$

The analytical solution for u is found by the same means as the previous equation. The phase portrait of Figure 3 shows an excellent convergence of the result. Unlikely, as shown in reference [7], strong numerical damping was introduced by some of the methods (Newmark with $\beta = 0{\cdot}3025$, $\gamma = 0{\cdot}6$; Wilson-θ with $\beta = 1/6$, $\gamma = \theta/2 = 1{\cdot}4$; Houbolt; α-method with $\alpha = -0{\cdot}1$, $\beta = 0{\cdot}3025$, $\gamma = 0{\cdot}6$) and with a rather small step $\Delta t = T_p/20 = 0{\cdot}0075$ s. It should be mentioned that the Newmark method with $\beta = 0{\cdot}25$ and $\gamma = 0{\cdot}5$ as well as with $\beta = 0$ and $\gamma = 0{\cdot}5$ [7] yields a phase plane similar to Figure 3. Then a similar plot was obtained using WEM but with a step 2·7 times larger.

TABLE 1

Maximum percentage error of WEM found with equation (28) *varying M and N. Solution of* $\ddot{v} + 100v(1 + 10v^2) = 0$, $v_0 = 1{\cdot}5$ and $\dot{v}_0 = 0$. *Time duration of the experiment* $T = 15$ s, $T/n = 0{\cdot}02$ s

M (terms in sequence)	N (partitions in numerical integration)			
	10	20	30	40
10	2·9	2·9	2·9	2·9
20	0·5	1·4	1·4	1·4
30	1·7	0·8	0·9	0·9
40	4·9	0·3	0·7	0·7

TABLE 2

Idem Table 1. $T/n = 0{\cdot}01$ s

M (terms in sequence)	N (partitions in numerical integration)			
	30	40	50	60
30	1·2	1·2	1·2	1·2
40	0·8	0·9	0·9	0·9
50	0·6	0·7	0·7	0·7
60	0·1	0·5	0·6	0·6

In order to observe a certain measure of WEM efficiency the maximum error percentage as a ratio of the energy difference

$$\text{Max. error} = \left|\frac{E - E_0}{E_0}\right| \times 100 \tag{28}$$

is depicted in Tables 1 and 2 for different values of M and N. The energy $2E = \dot{v}^2 + \bar{k}_1 v^2 + \bar{k}_2 v^4/2$ is calculated at each instant and E_0 using the same expression but at $t = 0$. Table 1 was calculated for $T/n = 0{\cdot}02$ s and Table 2 for $T/n = 0{\cdot}01$ s. As may be observed in the tables, the error converges to the right of the diagonal. This is due to the fact that M represents the number of terms in the extremizing sequence, i.e., the maximum number of semi-waves in the solution. On the other hand, N is the number of partitions in the numerical integration algorithm. Then it is necessary that $N \geqslant M$ in order to obtain acceptable results. The error remains unchanged in a row right of the diagonal. In effect, for a fixed value of M an increment of N (larger than M) does not improve the results.

The authors have found that for a fixed value of T/n (for instance $T/n = 0{\cdot}02$ s in Table 1) the increase of M above 40 (and N correspondingly) does not contribute towards lowering the error, i.e., the minimum error (0·7%) for $T/n = 0{\cdot}02$ s is attained within this methodology.

In Table 2, the value of T/n is halved. The same behavior is observed. Now the minimum error is 0·6% for $M \geqslant 60$. One concludes that the smaller the time step is, the larger M and N should be taken in order for the error to converge to a minimum.

TABLE 3

Maximum error. Comparison of different methods for $\ddot{v} + 100v(1 + 10v^2) = 0$, $v_0 = 1{\cdot}5$ *and* $\dot{v}_0 = 0$. *Time duration of the experiment* $T = 15$ s†

	Methods							
Time step	WEM	Average accel. [7]	Central diff. [7]	Newmark $\beta = 0{\cdot}305$ $\gamma = 0{\cdot}6$ [7]	Wilson [7]	Houbolt [7]	α [7]	Runge–Kutta [7]
$0{\cdot}01 = T_P/15$	3·7 (10, 10) 0·6 (60, 60)	7·1	7·7	99·8	93·9	98·8	67·8	44·5
$0{\cdot}0075 = T_P/20$	3·8 (10, 10) 0·5 (80, 80)	4·1	4·3	99·6	87·6	97·3	49·2	18·4

†*Note*: The numbers between parentheses in WEM denote (M, N).

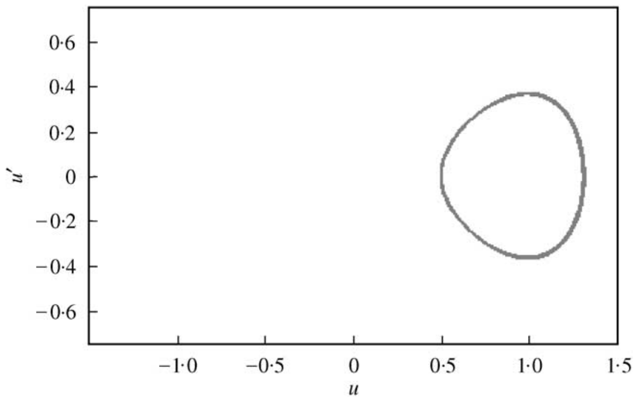

Figure 4. WEM solution of equation $\ddot{v} - 0{\cdot}5v(1 - v^2) = 0$, $v_0 = 0{\cdot}5$ and $\dot{v}_0 = 0$. Time duration of the experiment $T = 2500$ s. $M = 20$, $N = 10$, $T/n = 2{\cdot}5$ s.

Finally, Table 3 shows a comparison of the maximum error found using WEM with different values of M and N and the ones reported by Xie [7] obtained with the standard time integration techniques.

In general, various problems arise when large steps are used in numerical integration schemes. Let us solve equation (16) with $\bar{k}_1 = -0{\cdot}5$; $\bar{k}_2 = 0{\cdot}5$ with $v_0 = 0{\cdot}5$ and $\dot{v}_0 = 0$. The period is 7·3 s. As reported in reference [7], when the step was chosen to be $\Delta t = 2{\cdot}5\ \mathrm{s} \approx T_p/3$, the average acceleration method rendered a chaotic solution jumping between two static equilibrium points $v = 1$ and -1. Central difference and Runge–Kutta methods failed to yield a convergent solution and the other methodologies (Newmark, Wilson-θ, Houbolt and α- methods) introduced so much numerical damping that the results had no resemblance with the exact solution. In this hard test the WEM solution is reproduced in Figure 4 with an excellent agreement with the exact solution. It should be noted that the WEM solution with $M = 10$ was not convergent but $M = 20$ with a very small increment in the computational time gives the appropriate results.

5. APPLICATION OF WEM TO A NON-LINEAR PROBLEM GOVERNED BY A NON-HOMOGENEOUS PDE

Classically, the vibrational behavior of a beam subjected to dynamic forces is carried out after a separation of the spatial and time variables by means of, for example, Galerkin's method. That is, a proper set of approximating functions, which should satisfy all the boundary conditions—at least in the traditional approach—of the problem for all t is replaced in the differential equation. The residual error is then minimized. The result is an s.d.o.f. equation that may be solved by various means, according to the complexities involved.

Here the dynamics of a Bernoulli beam supported by non-linear springs at its ends and subjected to a load varying both in space and time is addressed with WEM. This vibration problem was also solved before using WEM but only regarding the space domain [15]. Here, instead, the two involved variables—space and time—are included in the WEM approach. Only a brief description is contained in this section. For more details see reference [16]. The space variable is non-dimensionalized with the length of the beam. Regarding the time variable, an interval of interest, T, is chosen. Although the problem is governed by a PDE with IC and BC a suitable transformation yields a BV problem in a two-dimensional domain.

The dynamics of a uniform beam are governed by the following partial differential equation:

$$v'''' + a^2\bar{\bar{v}} = q(x, \tau),$$

$$IC \begin{cases} v(x, 0) = U_0(x), \\ \bar{v}(x, 0) = V_0(x), \end{cases} \qquad BC \begin{cases} v(0, \tau) = v(1, \tau) = 0, \\ v''(0, \tau) = \beta_0 f_0(v'(0, \tau)) \equiv \beta_0 f_0(\tau), \\ v''(1, \tau) = \beta_1 f_1(v'(1, \tau)) \equiv \beta_1 f_1(\tau), \end{cases} \tag{29}$$

where

$$a^2 \equiv \frac{\rho A L^4}{EIT^2}, \quad q = q(x, \tau) = q_0(x)\cos(\omega\, T\tau), \qquad (\cdot)' \equiv \frac{\partial(\cdot)}{\partial x}, \quad \overline{(\cdot)} \equiv \frac{\partial(\cdot)}{\partial \tau}, \tag{30}$$

in which x is the non-dimensional space variable ($x = X/L, 0 \leqslant x \leqslant 1$), L is the length of the beam, τ is the non-dimensional time ($\tau = t/T, 0 \leqslant \tau \leqslant 1$), T is a time interval of interest, E is the modulus of elasticity, I is the moment of inertia of the cross-section of the beam, ρ is the mass density, A is the cross-section, $q(x, \tau)$ is the load (square integrable, it may be discontinuous both in space and time), β_0 and β_1 are spring constants f_0 and f_1 analytical functions of the slope (in general, non-linear) and $v = v(x, \tau)$ is the transverse displacement.

Let us now introduce $\psi = \psi(x, \tau)$ as in equation (8) and the new function $w = w(x, \tau)$ (recall equation (9)). With these definitions and denoting $p = p(x, \tau) \equiv q - \psi''''$, PDE (29) can be written as

$$w'''' + a^2\bar{\bar{w}} - p = 0 \tag{31}$$

with conditions

$$w(x, 0) = w(x, 1) = w(0, \tau) = w(1, \tau) = 0. \tag{32}$$

Thus, a boundary-value problem in two dimensions is obtained. To apply WEM we should introduce an extremizing sequence as stated in equation (13). Due to conditions (32) it yields

$$w_{MP}(x, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{P} A_{ij} s_i s_j. \tag{33}$$

Consequently, statement (12) yields

$$(w''_{MP}, \delta w''_{MP}) - a^2(\bar{w}_{MP}, \delta \bar{w}_{MP}) - (p, \delta w_{MP}) + \beta_0(f_0, \delta w'_{MP}(0, \tau))_\tau + \beta_1(f_1, \delta w'_{MP}(1, \tau))_\tau = 0. \tag{34}$$

Notation (F, G) stands for double integration. Instead the parentheses with subscript τ denote integration w.r.t. this variable. The following expression is obtained for the unknowns A_{ij}:

$$A_{ij} = \frac{4}{D_{ij}} \left\{ Q_{ij} - \frac{\alpha_i^4}{2}(\gamma_i I_{1j} + q_i I_{0j}) - \alpha_i(\beta_0 \lambda_{0j} + (-1)^i \beta_1 \lambda_{1j}) \right\}, \tag{35}$$

where

$$D_{ij} \equiv \alpha_i^4 - a^2 \alpha_j^2, \qquad Q_{ij} \equiv (q, s_i s_j), \qquad U_0(x) = \sum_i q_i s_i, \quad U_1(x) = \sum_i \gamma_i s_i,$$

$$I_{nj} \equiv (\tau^n, s_j)_\tau, \qquad \lambda_{nj} \equiv (f_n, s_j)_\tau, \quad i = 1, 2, \ldots, M, \; j = 1, 2, \ldots, P \;\; n = 0, 1. \tag{36}$$

Recall that $U_1(x)$ (unknown) may be eliminated from the equations stating the IC related to the first time derivative, as mentioned after expression (11).

5.1. NUMERICAL EXAMPLE

Let us assume a beam carrying a load $q(x, \tau) = q_0 x \cos(\omega T \tau)$ distributed along the length L of the beam and Duffing-type end springs, $f_0(\cdot) = f_1(\cdot) = (\cdot)^3$. Also, let us suppose

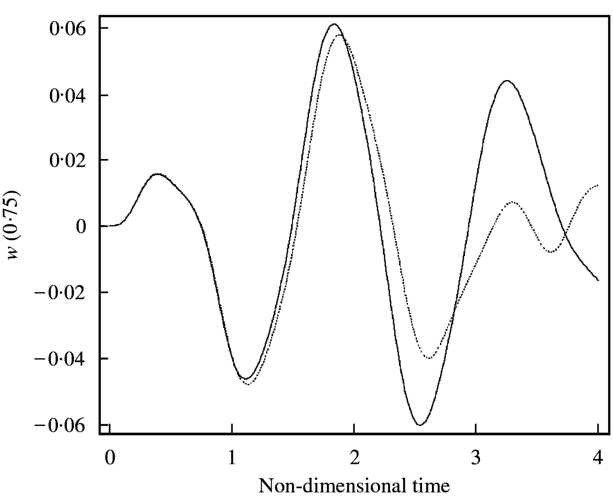

Figure 5. Displacement of the beam at $x = 0{\cdot}75$ as a function of $\tau = t/T$. $T = 4$ s, $T/n = 4$ s. $M = P = 20$, $N = 400$, $q_0 = 10$, $\omega = 5$ rad/s; $a^2 = 10/T^2$; $\beta_0 = \beta_1 = 15$. ——, Beam with non-linear springs; ··········, beam without springs.

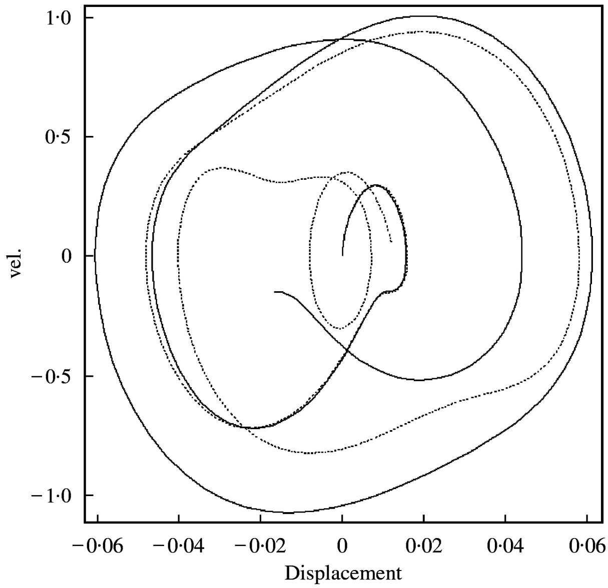

Figure 6. Idem Figure 5. Phase diagram of the beam at $x = 0{\cdot}75$.

that the system starts its movement undeformed and at rest, i.e., $U_0(x) = V_0(x) \equiv 0$. Using the above-described procedure the results may be plotted as shown in Figures 5 and 6. N is the number of steps for the Simpson algorithm.

It was verified that the solution of the beam without springs is coincident with the superposition modal solution. The authors find that the problem of non-linear boundary conditions would not be easily accessible by many methodologies. The present method permits its study in a systematic way and the adjustment of various parameters such as M, N and T/n, allows for a convergence of the solution.

6. CONCLUSIONS

An alternative analytic-numerical method based on a generalized solution and named WEM is used in the present work to solve ordinary and partial differential equations governing the behavior of finite d.o.f. or distributed infinite d.o.f. systems. It was originally developed by the authors to solve boundary value problems and afterwards extended to initial condition and mixed problems. Theorems and corollaries previously demonstrated assert the uniform convergence of the results. The method applies to linear or non-linear problems. A *pseudo* virtual work stated in certain types of extended series is the main feature of the technique. Both spatial as well as temporal variables are dealt with in a similar fashion. When dealing with time the method may be seen as a time integration scheme.

In the first part of the work, an application to the solution of a pair of non-linear ordinary differential equations is shown and some examples are solved numerically. A very good performance of the method is shown even when using time steps larger than the usual ones. A previous work of Xie [7] permits a comparison with other time integration techniques. WEM may be adjusted to yield arbitrary precision results. This is done by increasing the number of terms in the series (M). Another adjusting parameter is N (numerical integration terms). $N = 10$ was sufficient to attain acceptable results in the problems governed by ODE. Additionally, a measure of the error is computed to assess the efficiency of the method.

The third application is the solution of a non-linear partial differential equation. WEM is used to find the dynamic displacement of a beam subjected to a load varying, in general, with x (space) and t (time) and supported by non-linear springs. The adjustment of M, P, N and T/n permits the convergence of the solution and the possibility of finding arbitrary precision results.

ACKNOWLEDGMENTS

The first author is a scientific researcher of CONICET (Argentina). This work has been partially supported by a PGI grant from SGCyT (Universidad Nacional del Sur) and a PIP from CONICET. The authors acknowledge the reviewers for their suggestions and criticisms.

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